#### **4. Symbiosis model. Change of variables**

**4.1. Introduction**

**Object** Coexistence of two biological species

**Object clarification** coexistence ++, that is the the symbiosis model

**Foundation** Phase plane for the system of differential equations with the change of variables

**Aim** Definition of the law of change in the number of both species depending on the conditions of the process

**4.2. Symbiosis model**

**General supposition** There is a positive influence of both species on each other.

 The function *xi=xi*(*t*) describes the number of *i-*th species at the time *t*, *i=*1,2.

Consider the interaction of two species in symbiotic conditions, when each of the species has a beneficial effect on the second species. It is assumed that the *i*-th species dies out in the absence of another species with a certain speed *εi*, *i*= 1,2. At the same time, the greater the abundance of another species, the higher the growth of this species. Denoting by *γi* the coefficient of influence of the *j*-th type on the *i*-th one, we define the following type of growth abundance *ki* =-*i**ixj*, *i*≠*j*. As a result, the mathematical model of the system under consideration is characterized by differential equations

  (1)

We have the ***system of differential equations***.

For obtaining its uniqie solution, it is necessary to add the initial conditions. Suppose we now the initial numbers of species *x*10 and *x*20. Now we have theinitial conditions

 *xi*(0) = *xi*0, *i=*1,2. (2)

The Caushy problem (1), (2) id called the ***symbiosis model***. This model has six parameters that are four coefficients of the equations (1) and two initial states.

**4.3. Change of variables**

Try to make a change of variables so that the result is a problem with a smaller number of system parameters. Define the variables *τ*=*at*, *u*(*τ*)*=bx*1(*t*), *v*(*τ*)=*cx*2(*t*), where the constants *a*, *b* and *c* are chosen such that the resulting equations were as simple as possible.

Determine the derivative



because of the form of the variable *u* and *τ*. Using the first equality (1), we get



Hence, the first equality (1) takes the form

  (3)

Now we calculate the derivative



because of the form of the variable *v* and *τ*. Using the second equality (1), we get



Hence, the second equality (1) takes the form

  (4)

Thus, our system (1) is transformed to the equality (3), (4). Choose the constants *a*, *b* and *c* such that the resulting equations are as simple as possible. Particularly, we can choose the constants *b* and *c* sich that ther coefficient before the functions *u* and *v* under the brackets are equal to 1. This is true, if

  (5)

Now, the equalities (3), (4) are transformed to the system

  (6)

We have extra the free parameter *a* here. Choose it by the formula *a=*1 for obtaining the difference *v–*1 at the first equality (6). Therefore, we found all parameters for transformation the variables

  (7)

Now the system (6) takes the form

  (8)

where *m=*2/**1. The corresponding initial conditions are

 *u*(0) = *u*0, *v*(0) = *v*0, (9)

where *u*0*=γ*2*x*10/**1, *v*0*=γ*1*x*20/**1.

The system (8), (9) is the symbiosis model in the new coordinate system. It has the same properties as the intial system (1), (2). However, it depends on three parametars only that is the coefficient *m* of the second equation (8) and two unitial states of the equalities (9).

 **4.4. Equilibrium state for the system**

Find the equilibrium position for the system (8). Equating the right-hand sides of these equations to zero, we have two equalities

 (*v–*1)*u =* 0, (*u–m*)*v =* 0. (10)

From first of them, it follows that *v=*1 or *u=*0. Suppose the first equality is true. Then the second equality (10) takes the form *u–m=*0, so *u=m*. If *u=*0, then from second equality (10), we conclude *v=*0. Therefore, the system (8) has two equilibrium position. First of them is trivial *u=*0, *v=*0. The second one is non-trivial *u=*1, *v=m*.

**4.5. Dividing of the phase space for the system**

We consider again the first quadrant, because both state functions are non-negative as numbers of populations. Then the value at the right hand-side of the first equality (8) is positive if *v*>1, and this is negative if *v*<1. Analogically, the value at the right hand-side the second equality (8) is positive if *u*>*m*, and this is negative if *u*<*m*. Hence, the phase plane can be divided by four parts, see Figure 1. If *u*>*m* and *v*>1, then both derivatives are positive, so both functions increase. If *u*<*m* and *v*<1, then both derivatives are negative, so both functions decrease. Then, if *u*>*m* and *v*<1, then the function *u* decreases, and the function *v* increases. Finally, for the inverse situation, the function *u* increases, and the function *v* decreases.



Figure 1. Directions of the system evolution for the symbiosis model.

**4.6 Stability of the equilibrium position.**

Suppose both initial states are small, so they are in the small enough neighbourhood of the trivial equilibrium position. Hence, we are in the rectangle, where both derivative are negative, and both functions decrease. After decreasing of these functions, its derivatives stay negative as before; and the functions will decrease. Therefore, they tend to zero that is the trivial equilibrium position; see Figure 2. We conclude that this equilibrium position is stable.



Figure 2. Directions of the system evolution for the symbiosis model.

Now suppose the initial states belong to the small neighbourhood of the trivial equilibrium state that is the point (0,0). Particularly, they can satisfy the inequalities *u*0<*m* and *v*0<1. Hence, both functions decrease and do not tend to the equilibrium non-trivial state (*m*,1); see Figure 2. Besides, the initial state (*u*0,*v*0)can be close enough to (*m*,1), but satisfies the inequalities *u*0>*m* and *v*0>1. In this case, both functions increase and do not tend to the equilibrium state too; see Figure 2. Therefore, the non-trivial equilibrium position is non-stable.

**4.6. Analysis of the system evolution**

Suppose both initial states are small enough such that the following inequalities hold

 *u*<*m*, *v*<1 (11)

Then, in accordance with equalities (8), the derivatives of both functions are negative, which means that the functions themselves decrease. Thus, in subsequent time instants, inequalities (11) will be fulfilled with even greater justification. Consequently, derivatives will remain negative. Thus, over time, the functions will steadily decrease and tend to zero, i.e. to a stable equilibrium position; see Figure 3, curve 1.

Let, on the contrary, both initial values be so large that the conditions

 *u*>*m*, *v*>1 (12)

Then, according to equalities (8), the derivatives of both state functions are positive, which means that the functions themselves increase; see Figure 1. Consequently, conditions (13) will also be satisfied at subsequent instants of time, which will lead to an increase in the derivatives of the function under consideration. Thus, we observe their exponential growth; see Figure 3, curve 2.

Suppose one of the functions may turn out to be large enough and the other small enough so that the inequalities

 *u*>*m*, *v*<1 (13)

Then the derivative of the first of the state functions is negative, and the derivative of the second of them is positive. Thus, the first function decreases, and the second increases. This happens as long as conditions (13) are satisfied; see Figure 1. Further, two possible events are possible. Perhaps the function *u*, decreasing, becomes smaller than *m*,while the function *v*, increasing, has not yet reached the value 1. Thus, conditions (11) will be satisfied, which means that both functions will tend to zero; see Figure 3, curve 3. However, another situation is also possible when the function *v*, increasing, exceeds the value 1 earlier than the function *u*, decreasing, reaches a value *m*. Thus, conditions (12) are valid, which means that both functions increase unlimitedly; see Figure 3, curve 4.



Figure 3. Phase curves for the symbiosis model.

If the first of the considered functions is sufficiently small, and the second is sufficiently large, i.e. since both inequalities (13) are replaced by opposite ones, a picture similar to the previous one is observed. In this case, the first and second functions are interchanged; see Figure 3, curves 5, 6. In principle, one can prove, that there exists two phase curves, by which the system tends to the non-stable equilibrium position; see Figure 3, curves 7, 8.

**4.7. Interpretation of results**

The results obtained can be given the following interpretation. The small initial abundance values ​​of both species correspond to the extinction of each of them, since the available number of individuals in this population is not enough to support another population. In the presence of sufficiently large values ​​of species abundance, both of them are in favorable conditions. Therefore, in the absence of any limiting factors, the abundance of both species grows unlimitedly. If, initially, the abundance of one of the species is relatively large, and the abundance of another species is quite small, then the first of the species is in unfavorable conditions, and the second in favorable. As a result, the abundance of the first type is decreasing, and the second is growing. Two possible outcomes. If the abundance of the first species decreases earlier than the abundance of the second species, then over time both species will find themselves in unfavorable conditions, and the entire population will die out. Otherwise, the population is growing unlimitedly.